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DIMENSIONAL REDUCTION FOR FERMIONS*

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ABSTRACT

We generalize the concept of dimensional reduction to operators involving fermion fields in high temperature field theories. It is found that the ultraviolet behavior of the running coupling constant plays a crucial role. The general concept is illustrated explicitly in the Gross-Neveu model.

1. Introduction

Physics very often simplifies under certain extreme situations. For example, specific observables in a $D+1$-dimensional theory at high temperature (T) can sometimes be described by a D-dimensional theory: this phenomenon is known as dimensional reduction (DR). The basic concept of DR is that in the high-T limit all spatial excitations are naturally $\mathcal{O}(T)$. If, for either kinematic or dynamical reasons, there exist modes of order less than T, these few light modes may be the only active ones, while the others decouple. This qualitative expectation can be formalized[1] in some theories, such as QED, order by order in perturbation theory, analogously to the usual heavy mass decoupling.

The approach to DR is easier, when there exists a clear scale separation already at the tree-level. For instance, kinematics makes this scale hierarchy manifest for observables made by elementary boson fields. The non-zero Matsubara frequencies act like masses of $\mathcal{O}(T)$ compared to the zero-modes, and the heavy non-zero modes, both bosonic and fermionic, can be integrated out. If no other dynamical phenomena occur, the result is an effective theory in one dimension less, which can be used to describe static phenomena of the original theory in the high-T limit. So far, the existing literature has exclusively dealt with the dynamics of these bosonic zero modes.

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This work will focus on the situations where the observables are made explicitly by fermion fields, and, therefore, the lowest modes are also of $\mathcal{O}(T)$, and there is no obvious kinematic scale separation. If at all, the scale separation must be generated dynamically. We shall systematically study, in the frame work of perturbation theory and renormalization group, how the concept of DR can be generalized to include these cases. Our result will provide a formal basis for the recent interpretation of lattice data related to screening processes at high-T,[2–4] which assumes a picture where only the lowest Matsubara frequencies are important and can be treated non-relativistically. After stating the precise criterion for DR for observables involving fermion fields, we illustrate the general principle by explicitly calculating the screening mass in the Gross-Neveu model.

2. Dimensional reduction

Let us first recapitulate the criterion for DR when only static fundamental bosons appear in the external lines. If Green's functions with small external momenta ($|\mathbf{p}| \ll T$) in a theory described by the $D+1$ dimensional Lagrangian $\mathcal{L}_{D+1}$ are equal to the corresponding Green's functions of some specific D dimensional Lagrangian $\mathcal{L}_D$, up to corrections of the order $|\mathbf{p}|/T$ and m/T, where m is any external dimensionful parameter in $\mathcal{L}_{D+1}$, e.g. a mass, we say that DR occur for $\mathcal{L}_{D+1}$ with $\mathcal{L}_D$ as the effective theory. In general, the form and parameters of $\mathcal{L}_D$ are determined by the original theory. As stressed by Landsman,[5] this expectation may fail if there are dynamically generated scales of the order T. Nevertheless, these dynamically generated scales must be proportional to some power of the coupling constant $M \sim g^n T$, since they are generated by the interaction. Therefore, they induce corrections of order $M/T \sim g^n$. If g is small, the concept is still useful, and we say that the reduction is partial.

The case when composite operators made by fermions appear in the external lines is of great phenomenological interest, and includes external currents made out of fundamental fermionic fields such as the electromagnetic current, or mesonic interpolating fields in QCD. When fermions appear in external lines there are two main differences with the fundamental bosons case. The first difference is that the lowest Matsubara frequencies for fermions, $\omega_\pm = \pm\pi T$, are also the order T, and hence it is not obvious that they dominate over the heavier modes. The second difference is that we need to consider external momenta of order T. In fact, if we want the fermions to be close to their mass shell in the reduced theory (this is eventually the physically relevant region), $|\mathbf{p}|$ must be the order T.

Since $\omega_\pm$ acts in the reduced theory as a large mass, it has been proposed[2–4] that fermions might undergo a non-relativistic kind of dimensional reduction. This motivates us to define $q^2 \equiv \mathbf{p}^2 - (\pi T)^2$, and expand the Green's functions in the dynamical residual momentum q^2. In the end, it will be necessary to check the consistency of this expansion by verifying whether $q^2 \ll (\pi T)^2$. Again, we expect corrections of order m/T and q/T, and talk of partial reduction if we find that m and/or q is proportional to $g^n T$, with g small.

In analogy with the heavy mass decoupling theorem, the decoupling of the heavier modes is manifest only in specific subtraction schemes, such as the BPHZ scheme. A two-step approach better illustrates the need for a judicious choice of the counterterms.

Let us consider a graph in the original theory renormalized in a T-independent scheme, e.g. the $\overline{\mathrm{MS}}$ scheme. We can always split it into light and heavy contributions: $G^{D+1}(q,T) = G_L^D(q,T) + G_H^{D+1}(q,T)$, where G_L^D is the contribution of terms where *all* loop frequencies have their smallest value. Since there is no infinite frequency sums G_L^D is actually D dimensional. Then, we expand G_H at $q=0$ (the leading infrared behavior is contained in G_L by construction), and keep terms that are not suppressed by powers of T: $G^{D+1}(q,T) = G_L^D(q,T) + G_H^{D+1}(0,T) + O(q/T)$, where we have supposed that only the first term in the expansion survives. When DR takes place, the local term $G_H^{D+1}(0,T)$ can be eliminated by changing the renormalization prescription (i.e. adding finite counterterms), and we are left with the reduced graph G_L^D.

Because of the T-dependent renormalization, the parameters in the reduced graph (and in the effective Lagrangian that generates such graph) necessarily depend on T. This dependence is determined uniquely by the original theory.

3. A model calculation

Let us consider a concrete example, the Gross-Neveu model in 1+1 dimensions described by the Lagrangian

$$\mathcal{L} = \bar\psi i\gamma\cdot\partial\psi - \bar\psi(\sigma + i\pi\gamma_5)\psi - \frac{N}{2g^2}(\sigma^2+\pi^2)\,, \tag{1}$$

where σ and π are the auxiliary scalar and pseudoscalar boson fields respectively. We study this model in the limit $N\to\infty$ with the coupling constant g^2 fixed. By Fourier transforming the fields

$$\psi(\tau,x) = \sqrt{T}\sum_{n=-\infty}^{\infty}\psi_n(x)e^{i\omega_n\tau}\,, \qquad \begin{matrix}\sigma(\tau,x)\\ \pi(\tau,x)\end{matrix} = \sum_{l=-\infty}^{\infty}\begin{matrix}\sigma_l(x)\\ \pi_l(x)\end{matrix}\; e^{i\Omega_l\tau}\,, \tag{2}$$

where $\omega_n = (2n-1)\pi T$ and $\Omega_l = 2l\pi T$, we can rewrite the action in the following way

$$\int dx\left\{\sum_{n,l=-\infty}^{\infty}\bar\psi_n(x)\Big[-\omega_n\gamma_0 - i\gamma_1\partial_1 - \sigma_l(x) + i\gamma_5\pi_l(x)\Big]\psi_n(x) - \frac{N}{2g^2T}\Big[\sigma_l^2(x)+\pi_l^2(x)\Big]\right\}. \tag{3}$$

Since we are only interested in Green's functions with zero external frequency, only terms with $l=0$ are relevant. We are left with a one dimensional fermion theory of infinite species, each with a chirally invariant masses ω_n. The tree-level coupling constant is g^2T. We expect that DR occurs, if the static correlations are reproduced by the action (3) with only $n=\pm1$, and the higher modes at most modify the tree-level effective theory in a local way, i.e. make the parameters T-dependent.

In the $1/N$ limit and high-T regime, this model has only one non-trivial irreducible graph: the bubble graph. This graph in the high-T phase for static external lines is[6]

$$\begin{aligned} i\Pi(p_1) &= -iNT\sum_{n=-\infty}^{\infty}\mu^{2\epsilon}\int\frac{d^{1-2\epsilon}k_1}{(2\pi)^{1-2\epsilon}}\,\mathrm{Tr}\left\{\gamma_5\frac{i}{k\cdot\gamma}\gamma_5\frac{i}{(k+p)\cdot\gamma}\right\} && (4)\\ &= -\frac{N}{2\pi}\left[\frac{1}{\epsilon} + \ln\left(\frac{4\mu^2\,e^{\gamma_E}}{\pi T^2}\right) - \frac{p_1^2}{T^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3\pi^2 + (2n-1)p_1^2/4T^2}\right]. && (5)\end{aligned}$$

Let us renormalize the theory in a T-independent way. For instance, the $\overline{\mathrm{MS}}$ scheme yields the renormalized coupling $2\pi/g^2(\mu) = \ln(4\mu^2 e^{\gamma_E}/\pi T_c^2)$ where $T_c \equiv \Lambda_{\overline{MS}} e^{\gamma_E}/\pi$. Since we expect that the large distance behavior of the spatial correlations is determined by the lowest singularities, i.e. $p_1 = \pm 2\pi T$, we analytically continue p_1^2 into Minkowski space, and define the reduced momentum $q_1^2 = -p_1^2 - 4\pi^2 T^2$. Then Eq. (5) becomes

$$\begin{aligned} i\Pi(q_1^2) &= -\frac{N}{\pi}\ln(\frac{T_c}{T}) - \frac{2N}{\pi}\frac{4\pi^2T^2 - q_1^2}{q_1^2}\left[1 + \sum_{n=1}^{\infty}\frac{q_1^2}{(2n+1)[16\pi^2T^2(n^2+n) - q_1^2]}\right] \\ &= -\frac{N}{\pi}\ln(\frac{T_c}{T}) - \frac{2N}{\pi}\frac{4\pi^2T^2}{q_1^2}\left[1 + \mathcal{O}(\frac{q_1^2}{T^2})\right]. \end{aligned} \tag{6}$$

This result illustrates what has been said in the previous section. The main momentum dependence $-(2N/\pi)(4\pi^2T^2/q_1^2)$ is given by the lightest modes. The other modes give contributions that are either suppressed by powers of q_1^2/T^2, or momentum independent, $-N\ln(T_c/T)/\pi$. But this local piece can be eliminated by a judicious choice of renormalization scale μ, e.g. $4\mu^2 = \pi T^2 e^{-\gamma_E}$. Consequently, the coupling that appears in any diagram of the original theory will effectively depend on T through:

$$g^2\,(\mu)|_{4\mu^2 e^{\gamma_E} = \pi T^2} = \frac{\pi}{\ln(T/T_c)} \equiv g^2(T)\,. \tag{7}$$

Obviously, the same T-dependence has to be inherited by the reduced theory. One can explicitly check that this graph can be generated from the reduced action, Eq. (3) with $l = 0$ and $n = \pm 1$, with that g^2 replaced by $g^2(T)$ given by Eq. (7).

In the original theory solving the equation $N/g^2 - i\Pi = 0$ yields the screening mass

$$\tilde{m} = 2\pi T\left[1 - \frac{1}{\pi}g^2(T) + \mathcal{O}(g^4(T))\right]. \tag{8}$$

This result is reproduced by the reduced theory to leading order in $g^2(T)$, and explains the physical reason why a partial DR works for this theory. The screening state is a bound state of a quark and an antiquark of mass πT, with a binding energy $\approx 2g^2(T)T$. The binding energy in units of the quark mass decreases logarithmically, so we expect that the screening mass can be solved from an appropriate Schrödinger equation in the high-T limit. The point here is that there exists a simplified physical picture, similar to the non-relativistic reduction assumed by several authors.[2–4] In an asymptotically free theory, there is one more scale appearing other than T. This new scale, $T/\ln T$, makes the partial DR possible. On the contrary, in theories with a finite ultraviolet fix-point, as we have explicitly verified in the 2+1 Gross-Neveu model,[7] not even this partial dimensional reduction occurs, due to the lack of the scale hierarchy.

4. Conclusions and outlook

We generalized the usual DR concept to the case of operators made by fermion fields, where there is no obvious scale separation. Then, the idea is illustrated in the Gross-Neveu model by solving the screening mass in the high-T limit. The experience gained in the model study suggests that the complete DR (with corrections suppressed

by powers of T) in case of operators made by fermion fields is almost impossible, since the overall scale is set by T. However, when the theory is asymptotically free a new scale, $g^2(T)T$, becomes available, which makes the partial DR possible.

Although the partial DR is the most likely scenario for operators involving fermion fields in asymptotically free theories, the partial reduction would still be sufficient to guarantee a simplified physical picture, the non-relativistic type of reduction, and thus is useful to interpret lattice data at high-T.[2–4] We are in the process of explicitly examining whether a similar partial DR is going to occur for currents made by fermions in QED and QCD.

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